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Irreducible Triangulations of Surfaces

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1. INTRODUCTION

In this note we show that, for any surface Σ and any k, there are at most finitely many triangulations of Σ such that each edge is in a noncontractible cycle of length k and is in no shorter noncontractible cycle. Such a triangulation is k-*irreducible*. This is equivalent to the statement that for any surface Σ and any k, there are at most finitely many embeddings in Σ that are minor minimal with representativity k.

This last fact can be derived from a theorem (a variant of Wagner's conjecture) that graphs embedded in a surface, with vertices and edges labelled from a well-quasi-order, form a well-quasi-order under abstract minors respecting the labels. However, this proof is very complicated and is not constructive. Thus, it is desirable to have an elementary proof of this particular consequence.

Recently, several papers have dealt with the problem of showing that there are at most finitely many 3-irreducible triangulations [BE, GRT, NO]. Malnič and Mohar [MM] prove that there are at most finitely many 4-irreducible triangulations of an orientable surface. Malnič and Nedela [MN] have given the first elementary proof that the number of k-irreducible triangulations of Σ is finite for all k and all Σ. Gao *et al.* [GRT] have a very simple proof that there are at most $c\xi^4$ vertices in a 3-irreducible triangulation of any surface (orientable or not) with Euler characteristic $2-\xi$, while Nakamota and Ota [NO] show (with a similar simple proof) that in fact there are at most $c\xi$ vertices in such a triangulation.

In this note, we give a very short, simple proof of the Malnič and Nedela theorem. Moreover, we give an explicit estimate of the form $c_k\xi^2$ on the

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size of a k-irreducible triangulation. For $k=3$, this is not as good as the bound of Nakamota and Ota.

For a surface Σ, the *Euler genus* of Σ is the number $\xi(\Sigma)=2-\chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic of Σ. The Euler genus is $2h$ if Σ is the sphere with h handles and k if Σ is the sphere with k crosscaps.

MAIN THEOREM. *Let T be a k-irreducible triangulation of Σ, for some $k \geqslant 2$. If Σ has Euler genus ξ, then*

$$|E(T)| \leqslant 3k \cdot k!\,(6k)^k\,\xi^2.$$

We expect that the correct estimate for $|E(T)|$ in the Main Theorem is $c_k \xi$ rather than $c_k \xi^2$. Previously, the bounds on the size of the irreducible triangulations have been in terms of $|V(T)|$, rather than $|E(T)|$. Since Euler's formula shows $|V(T)| = (1/3)\,|E(T)| + 2 - \xi$, there is really no difference. In this work, it is more convenient to use $|E(T)|$.

In order to prove this theorem, we require some preliminary results. The central idea we need is that of a crossing of two cycles in an embedded graph. So let C_1 and C_2 be distinct but not disjoint cycles in a graph embedded in a surface Σ. Let P be a component of $C_1 \cap C_2$, so that P is a path, possibly consisting of just a vertex. If P is more than just a vertex, there is an open disc Δ in Σ with $P \subset \Delta$ and a homeomorphism $h\colon \Delta \to \mathbb{R}^2$ such that $h(\Delta \cap (C_1 \cup C_2))$ consists of the segment from -1 to 1 on the x-axis ($h(P)$) and four vertical rays, two emanating from the point $(-1, 0)$ and two from $(1, 0)$. (See [HR, R] for similar arguments.) The path P is a *crossing of* C_1 *and* C_2 if the ray down from $(-1, 0)$ and the ray up from $(1, 0)$ are contained in the same one of $h(C_1 \cap \Delta)$ and $h(C_2 \cap \Delta)$. If P is just a vertex, then $h(P)$ is the origin and the four rays are contained in the two lines $L_1: y=x$ and $L_2: y=-x$. It is a crossing if h can be chosen so that $h(C_1 \cap \Delta)=L_1$ and $h(C_2 \cap \Delta)=L_2$.

If C_1 and C_2 are distinct cycles and P is a component of $C_1 \cap C_2$, then P is the *v-component of intersection* if $v \in V(P)$.

A *k-edge-width* embedded graph G in a surface Σ that is not the sphere is an embedded graph in which no noncontractible cycle in G has length less than k. Obviously, if a closed walk W in a k-edge-width embedding G is noncontractible, then W has length at least k and if the length of W is exactly k, then W is a cycle.

In what follows, there are two types of homotopies that we use. Let Σ be a topological space and let $\gamma_1, \gamma_2 : [0, 1] \to \Sigma$ be curves (i.e. continuous functions). They are *homotopic with fixed endpoints* if there is a continuous function $H\colon [0, 1] \times [0, 1] \to \Sigma$ such that: (1) for each $s \in [0, 1]$, $H(s, 0) = \gamma_1(s)$ and $H(s, 1) = \gamma_2(s)$; and (2) for all $t \in [0, 1]$, $H(0, t) = \gamma_1(0) = \gamma_2(0)$ and $H(1, t) = \gamma_1(1) = \gamma_2(1)$.

Now suppose γ_1 and γ_2 are both closed, i.e. For $i=1, 2$, $\gamma_i(0)=\gamma_i(1)$. Then they are *homotopic* if there is a continuous function $H: [0, 1] \times [0, 1] \to \Sigma$ such that: (1) for each $s \in [0, 1]$, $H(s, 0)=\gamma_1(s)$ and $H(s, 1)=\gamma_2(s)$; and (2) for all $t \in [0, 1]$, $H(0, t)=H(1, t)$.

Lemma 1. *Let G be a k-edge-width embedding in a surface Σ and let v be a vertex of G. Suppose that every edge of G is in a noncontractible k-cycle that contains v. Then there is a covering of G by noncontractible k-cycles, all containing v, such that no two of the covering k-cycles cross, except possibly at the v-component of intersection.*

Proof. Define a (k, v)-*noncontractible cycle cover* of G ((k, v)-ncc, for short) to be a set $\mathfrak{C}=\{C_1, ..., C_n\}$ of noncontractible k-cycles in G such that each C_i contains v and $G=\bigcup_{i=1}^n C_i$. One hypothesis of the theorem is there is a (k, v)-ncc of G.

Let $\mathfrak{C}=\{C_1, ..., C_n\}$ be a fixed (k, v)-ncc of G and create a new embedded graph G' as follows. We wish to create copies $C'_1, ..., C'_n$ of $C_1, ..., C_n$, respectively, which are pairwise edge-disjoint. This is done by replacing each edge e of G with parallel edges (having the same ends as e and drawn in a disc neighbourhood of e), one for each of the C_i that contains e. These parallel edges are then arbitrarily bijectively assigned to the cycles.

For two cycles Q, Q' of a (k, v)-ncc $\mathfrak{C}'$ of G', whose members are pairwise edge-disjoint, define $\mathrm{cr}_v(Q, Q')$ to be the set of crossing vertices in $C \cap C'$, excluding v. Now set $f_v(\mathfrak{C}')=\sum_{Q, Q' \in \mathfrak{C}'} |\mathrm{cr}_v(Q, Q')|$.

Let $\mathfrak{C}'=\{Q_1, Q_2, ..., Q_n\}$ be a (k, v)-ncc of G', whose members are pairwise edge-disjoint, such that $f_v(\mathfrak{C}')$ is minimized. We claim that $f_v(\mathfrak{C}')=0$. This is enough to prove Lemma 1, since collapsing the created parallel edges (and eliminating duplicate cycles) yields $\mathfrak{C}$, a (k, v)-ncc of G that satisfies the conclusion of the lemma. So we suppose $f_v(\mathfrak{C}')>0$ and derive a contradiction.

There is some pair i, j for which $\mathrm{cr}_v(Q_i, Q_j)$ has at least one member. Let x be a vertex in $\mathrm{cr}_v(Q_i, Q_j)$. For $l=i, j$, let W_l and W'_l be the two paths in Q_l from v to x and let w_l and w'_l be their lengths, respectively. We remark that $w_l+w'_l=k$.

Lemma 2. *Let C_1 and C_2 be noncontractible k-cycles in a k-edge-width embedding having the vertices u and v in common. For $i=1, 2$, suppose W_i and W'_i are the paths from u to v in C_i. Then either both $W_1 W_2^{-1}$ and $W'_1 W_2'^{-1}$ are noncontractible k-cycles or both $W_1 W_2'^{-1}$ and $W'_1 W_2^{-1}$ are noncontractible k-cycles. Moreover, C_1 and C_2 traverse their common vertices in the same cyclic order.*

Proof. If one of $W_1 W_2^{-1}$ and $W'_1 W_2'^{-1}$ is contractible and if one of $W_1 W_2'^{-1}$ and $W'_1 W_2^{-1}$ is contractible, then it is easily checked that either C_1 or C_2 is contractible, a contradiction. For sake of definiteness, we assume both $W_1 W_2'^{-1}$ and $W'_1 W_2^{-1}$ are noncontractible. These closed walks each have length at least k and total length exactly $2k$. Thus, each has length exactly k and is, therefore, a cycle.

For the moreover claim, let C_1 traverse the common vertices $u_1, u_2, ..., u_t$ in this cyclic order. Suppose that some u_i and u_j are consecutive in C_2, but not in C_1. Let W_2 be the path in C_2 that joins u_i and u_j but is otherwise disjoint from C_1. Let W'_2 be the path in C_2 such that $C_2 = W_2 W'_2$.

In each of the two paths in C_1 joining u_i and u_j, there is another u_k. But then W'_2 must intersect both the paths in C_1 joining u_i and u_j, contradicting the fact proved above that it is disjoint from at least one of them. ∎

The proof of Lemma 1 will be completed when we show that $f(\mathfrak{C}') = 0$. To do this we apply Lemma 2. We can choose the labelling so that both $W_j^{-1} W'_i$ and $W_j'^{-1} W_i$ are noncontractible k-cycles.

Let $\mathfrak{C}''$ be $\mathfrak{C}'$ with Q_i and Q_j replaced by $Q'_i = W_j^{-1} W'_i$ and $Q'_j = W_j'^{-1} W_i$. Clearly $\mathfrak{C}''$ consists of edge-disjoint noncontractible k-cycles that cover G' and all contain v. Thus, $\mathfrak{C}''$ is a (k, v)-ncc of G.

The contradiction will be complete when we show that $f_v(\mathfrak{C}'') < f_v(\mathfrak{C}')$. It is clear that $\mathrm{cr}_v(Q'_i, Q'_j) < \mathrm{cr}_v(Q_i, Q_j)$, so it suffices to show that, for any cycle $Q \in \mathfrak{C}' \setminus \{Q_i, Q_j\}$, $|\mathrm{cr}_v(Q, Q'_i)| + |\mathrm{cr}_v(Q, Q'_j)| \leqslant |\mathrm{cr}_v(Q, Q_i)| + |\mathrm{cr}_v(Q, Q_j)|$.

Let w be any vertex of G other than v that is in Q. If $w \neq x$, then it is clear that Q has exactly as many crossings at w with Q_i and Q_j as it does with Q'_i and Q'_j. If $w = x$, then the six edges incident with w in the cycles Q_i, Q_j, Q occur in one of the three (up to symmetry) cyclic orders: $(Q_i, Q, Q, Q_j, Q_i, Q_j)$, $(Q_i, Q, Q_j, Q, Q_i, Q_j)$, and $(Q_i, Q, Q_j, Q_i, Q, Q_j)$. In the first case Q does not cross any of Q_i, Q_j, Q'_i and Q'_j at w, in the second case it crosses Q_j and exactly one of Q'_i and Q'_j, while in the third case Q crosses both Q_i and Q_j and either neither or both of Q'_i and Q'_j. From these remarks we have $|\mathrm{cr}_v(Q, Q'_i)| + |\mathrm{cr}_v(Q, Q'_j)| \leqslant |\mathrm{cr}_v(Q, Q_i)| + |\mathrm{cr}_v(Q, Q_j)|$, as required. ∎

Let G be an embedded graph in a surface Σ and let v be a vertex of G. The cycles $C_1, ..., C_n$ in G are *v-noncrossing* if no two of the C_i cross, except possibly at the v-component of intersection, if it exists.

In what follows, we will have occasion to modify an embedding G that is the union of v-noncrossing cycles $C_1, ..., C_n$, all containing v, into an embedding in which the cycles C_i are pairwise disjoint except for v. We want to do this only by "small" changes that do not affect the order of the cycles at a vertex or the homotopy type of the cycles.

For this procedure, we can assume that no vertex of G is of degree 2, for we can suppress the degree 2 vertices. We can further assume that each of the closed curves $C_1, ..., C_n$ begins and ends with v.

Let w be a vertex different from v such that two of the C_i contain w. It is easy to see that in the rotation of edges around w there are two consecutive edges, say e and f, in the same C_i. We create a new graph embedding in the surface with new cycles $C'_1, C'_2, ..., C'_n$, homotopic to $C_1, C_2, ..., C_n$, respectively.

Split the vertex w into two vertices, one incident with all the edges currently incident with w and the other incident only with new copies of e and f (which are drawn in the angle between the original e and f). All the cycles that contain both e and f will contain the new copies instead. All the remaining cycles remain unchanged. If no cycle now goes through the original e or f, then any such edge should be deleted.

The components of $C_i \cap C_j$ are in 1–1 correspondence with the components of $C'_i \cap C'_j$, with the exception of the possibility that w was exactly a component of intersection and exactly one of C_i and C_j contains both e and f. Corresponding components are either both crossing or both non-crossing (because $w \neq v$).

Since w has degree at least 3, there are cycles C_i and C_j with C_i containing e and f and C_j containing some other edge incident with w. Thus, we have reduced the total number of intersections and repeating this procedure yields the desired disjoint cycles.

Lemma 3. *Let G be an embedded graph in a surface Σ with Euler genus $\xi > 0$ and let v be a vertex of G. Let $C_1, ..., C_n$ be cycles in G that all contain v, are noncontractible in Σ, are pairwise not homotopic in Σ and are v-noncrossing. Then $n \leqslant 3\xi - 3$, except when $\xi = 1$, in which case $n \leqslant 1$.*

This is essentially Proposition 3.6 in [MM]. In the nonorientable case, n can be as large as $3\xi - 3$, but in the orientable surface of genus g, $\xi = 2g$, while the upper bound of $3g - 3$ holds and can be attained (see Proposition 3.7 in [MM], a weaker version of which is Corollary 3.2 below).

Proof. By the modification procedure described in the preceding paragraphs, we can assume the C_i are pairwise disjoint except for v. Suppressing degree 2 vertices, we can further assume the C_i are loops. It is an elementary exercise to show that finitely many more loops can be added to G so that every face of the embedded graph is homeomorphic to an open disc while still retaining the property that no loop is contractible and no two loops are homotopic.

We suppose there are N loops and M faces, so that $n \leqslant N$. No face boundary has length 1, since then the face is bounded by a contractible loop.

If some face boundary has length two, then let L_1 and L_2 be the loops occurring in the boundary walk. If $L_1 \neq L_2$, then L_1 and L_2 are homotopic, a contradiction. Therefore, $L_1 = L_2$. Consider the rotation of edge-ends around v. The only corners that occur are L_1 with itself, so that L_1 is the only loop incident with v. It follows that $\xi = 1$ and $N = 1$. Thus, we can assume that no face boundary has length two.

Therefore, $3M \leqslant 2N$, so the lemma follows from Euler's formula $1 - N + M = 2 - \xi$. ∎

The *b-bouquet* is the graph with one vertex and b loops.

Corollary 3.1. *Suppose Σ is a surface with Euler genus $\xi > 0$. Suppose the b-bouquet is embedded in Σ so that no loop is the boundary of a closed disc and no pair of loops bounds an open disc. Then $b \leqslant 3\xi - 3$, unless $\xi = 1$, in which case $b \leqslant 1$.*

Corollary 3.2. *Suppose Σ is a surface with Euler genus $\xi > 0$. Suppose there are b pairwise disjoint, pairwise not homotopic noncontractible simple closed curves in Σ. Then $b \leqslant 3\xi - 3$, unless $\xi = 1$, in which case $b \leqslant 1$.*

Proof. In the surface there is a collection of simple paths which are pairwise disjoint from each other, disjoint, except for their endpoints, from the simple closed curves and connect up all the simple closed curves. Contracting these in an appropriate way yields an embedding of the b-bouquet in Σ with no loop contractible and no two loops homotopic. The result follows from Corollary 3.1. ∎

The *b-bond* is the graph with two vertices and b edges between them.

Lemma 4. *If a b-bond is embedded in a surface with Euler genus $\xi > 0$ so that no two edges bound a closed disc, then $b \leqslant 2\xi$.*

Proof. The proof is the same as the proof of Lemma 3, except now the graph has two vertices and is bipartite, so every face has at least four edges in its boundary. ∎

Our final homotopy result is the following.

Lemma 5. *Let G be a k-edge-width embedded graph in a surface Σ without parallel edges. Let $u_1, ..., u_t$ be vertices of G and let $C_1, ..., C_n$ be distinct noncontractible k-cycles of G, all containing the u_i (in this cyclic order) but otherwise pairwise vertex-disjoint. Suppose u_1 is the end of each C_i and that $C_1, ..., C_n$ are pairwise u_1-noncrossing and pairwise homotopic. Let t' be the number of i such that u_i is not adjacent in G to u_{i+1}. (Indices are modulo t.)*

(1) *If $t' > 2$, then there is an i such that there are n pairwise internally disjoint paths from u_i to u_{i+1} that are pairwise homotopic with fixed endpoints.*

(2) *If $t' = 1$ or 2, then there is an i such that there are n pairwise internally disjoint paths from u_i to u_{i+1} that break up into at most three distinct homotopy classes with fixed endpoints.*

(3) *If $t = 1$, then the n cycles break up into at most two distinct homotopy classes with u_1 fixed.*

We remark that Lemma 2 shows that the assumption about the C_i traversing all the u_j in the same cyclic order is not really necessary. However, for the statement, we did need to know that order.

Proof. Suppose first that $t \geqslant 2$. If one of the C_j contains an edge $u_i u_{i+1}$, then they all contain this edge, since a noncontractible k-cycle in a k-edge-width embedded graph without parallel edges has no chords. Since the cycles are distinct, there must be some i such that no C_j has an edge joining u_i and u_{i+1}.

The paths $P_1, ..., P_n$ in $C_1, ..., C_n$, respectively, from u_i to u_{i+1} are, from the hypotheses, pairwise internally disjoint. Suppose two of the P_i are not homotopic.

Without loss of generality, we may assume P_1 is not homotopic to P_2. Let Q_1 and Q_2 be the paths in C_1 and C_2, respectively, from u_{i+1} to u_i, so that $C_1 = P_1 Q_1$ and $C_2 = P_2 Q_2$. If $P_1^{-1} P_2$ were contractible, then, since it is a cycle, it would bound a disc. It would follow that P_1 and P_2 are homotopic with fixed ends. Therefore, $P_1 P_2^{-1}$ is not contractible.

It follows that $P_1 P_2^{-1}$ has length at least k. Suppose there is a $j \neq i$ such that the paths R_1 and R_2 in C_1 and C_2 from u_j to u_{j+1} are not homotopic with fixed ends. Then, for the same reasons, $R_1 R_2^{-1}$ has length at least k. Since, for $i = 1, 2$ P_i and R_i are subpaths of C_i, their total length is at most k. Thus, $P_1 P_2^{-1}$ and $R_1 R_2^{-1}$ have total length at most $2k$. Thus, if $t' \geqslant 2$ and $t > 2$, for each pair of cycles, there is at most one i for which the paths in that pair from u_i to u_{i+1} are not homotopic with fixed endpoints.

We continue with the case $t' \geqslant 2$ and $t > 2$. Now suppose there is an i and a j, $i \neq j$, such that the paths in C_1 and C_2 from u_i to u_{i+1} are not homotopic with fixed endpoints and the paths in C_3 and C_4 from u_j to u_{j+1} are not homotopic with fixed endpoints. We allow the possibility that one of C_3 and C_4 might equal C_1 or C_2.

The paths in C_3 and C_4 from u_i to u_{i+1} are homotopic with fixed endpoints from the earlier discussion. Thus, both are not homotopic with fixed endpoints to one of the paths in C_1 and C_2. But at least one of the paths in C_3 and C_4 from u_j to u_{j+1} is not homotopic with fixed endpoints to the (homotopic) paths in C_1 and C_2. This contradicts the earlier conclusion that two cycles can have non-homotopic paths with fixed endpoints in at most one place.

(We thank Richard Brunet for pointing out the need for and supplying the argument of the preceding paragraph.)

Thus, in the case $t' \geqslant 2$ and $t > 2$, we are done: there is an i such that the disjoint paths from u_i to u_{i+1} are pairwise homotopic with fixed ends.

If either $t' = 1$ or $t' = t = 2$, then we must work a little harder. Because the cycles cross at most only at u_1, we can use the modification procedure described before Lemma 3 to separate them slightly at all the other vertices to create cycles that are pairwise disjoint, except for u_1 and are homotopic to the original cycles.

If no two of the cycles cross at u_1, then we can also separate them slightly at u_1 to create totally disjoint cycles $C'_1, C'_2, ..., C'_n$ which are homotopic to the original cycles. As these disjoint cycles are pairwise homotopic, they pairwise bound cylinders. Thus, there are two, say C'_1 and C'_n, which bound a cylinder containing all the other C'_i. It follows that $C_1 \cup C_n$ bounds an open cylinder that contains all the other C_i.

Each C_i consists of paths joining the various u_j. There are most two paths of interest—the ones that do not occur in all the C_i. We can assume the first joins the vertices v_1 and v_2, while the second, if it exists, joins v_3 and v_4. If either $v_2 \neq v_3$ or $v_4 \neq v_1$, then the earlier argument shows that either all the paths from v_1 to v_2 are homotopic with fixed ends or all the paths from v_3 to v_4 are homotopic with fixed ends.

We show the disjoint paths from v_1 to v_2 break up into at most three homotopy types. In the cylinder bounded by C_1 and C_n, this path joins v_1 on one of the two boundary curves to v_2 on one of the two boundary curves. Each of the four possibilities can be a different homotopy type, depending on how the identifications of v_1 and v_2 as points in the cylinder are made to recapture a subset of the surface.

However, there are two paths, one starting at v_1 on the "left" boundary and ending at v_2 on the "right" boundary and the other starting at v_1 on the right and going to v_2 on the left. It is easy to see that we cannot complete both of these to homotopic cycles without some intersection in the interior of the cylinder. Thus, there are only at most three different homotopy types that occur among the paths. This settles the case $t' = 1$ or $t' = t = 2$ in the case the cycles are nowhere crossing.

If two of the C_i cross at u_1, then pairwise every two of the C_i must cross at u_1. (This is, for a fixed homotopy class of cycles, the parity of the number of crossings between pairs of cycles in the class is constant.) Separating the C_i at the other u_i yields pairwise homotopic cycles $C'_1, ..., C'_n$ that are disjoint except for u_1. Now any two of the C'_i bound a disc, so that there is some pair, say C'_1 and C'_n that bound a disc containing all the other C'_i.

Thus, C_1 and C_n bound an open disc containing all the other C_i. This disc has boundary $P_1 Q_1 Q_n^{-1} P_n^{-1}$, where P_i and Q_i are the subpaths of C_i from u_1 to u_2 and u_2 to u_1, respectively. The other cycles consist of two paths across the disc, from one copy of u_1 to one of the copies of u_2 and from the same copy of u_2 to the other of u_1. Thus, there are at most two different homotopy types for the P_i. This completes the proof in the case $t' = 1$ or $t' = t = 2$.

Finally, we assume $t = 1$, so that the cycles C_i are pairwise disjoint except for u_1. If they do not cross at u_1, then we get, as above, that two of them, say C_1 and C_n bound an open cylinder containing all the remaining cycles. In this case, each of the C_i is homotopic with u_1 fixed either to C_1 or to C_n, so there are at most two homotopy types. (We remark that two homotopic cycles can intersect in just one point and not be homotopic with the point fixed. See Fig. 1.)

If the cycles pairwise cross at u_1, then two of them, say C_1 and C_n bound an open disc containing all of the C_i. In this case, all the C_i are homotopic with u_1 fixed to C_1. ∎

LEMMA 6. *Suppose $\mathscr{S}$ is a set of $k!\,n^k$ distinct k-sets, $k \geqslant 1$. Then there is a set B and a subset $\mathscr{S}'$ of $\mathscr{S}$ consisting of n k-sets such that, for each $S, S' \in \mathscr{S}'$, $S \cap S' = B$.*

Proof. We proceed by induction on k. The result is trivial for $k = 1$, in which case $\mathscr{S}$ consists of n distinct singletons. Now suppose $k > 1$.

If there is some element s in at least $(k-1)!\,n^{k-1}$ sets in $\mathscr{S}$, then look at these, delete s, and apply the inductive assumption.

If there is no element s in at least $(k-1)!\,n^{k-1}$ sets in $\mathscr{S}$, then each set in $\mathscr{S}$ intersects fewer than $k!\,n^{k-1}$ sets in $\mathscr{S}$. In this case, there are at least $\{k!\,n^k\}/\{k!\,n^{k-1}\} = n$ pairwise disjoint sets in $\mathscr{S}$. ∎

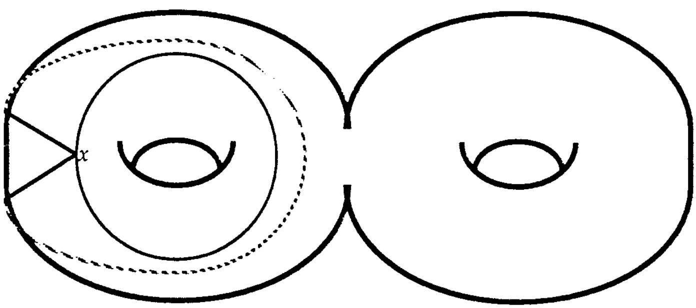

FIGURE 1

Proof of the Main Theorem. In what follows, we assume $k \geqslant 3$. The case $k=2$ has parallel edges, which requires a slight modification based on Lemma 4, which we omit. We suppose $|E(T)| \geqslant 3k \cdot k!\,(6k)^k\,\xi^2$ and derive a contradiction. Let $\mathfrak{C}$ be a covering of T by noncontractible k-cycles, such that $|\mathfrak{C}|$ is as small as possible. Clearly $|\mathfrak{C}| \geqslant |E(T)|/k \geqslant 3k!\,(k)^k\,\xi^2$.

First, suppose that some vertex v is in at least $3\xi(k-1)!\,(6k)^{k-1}$ of the cycles in $\mathfrak{C}$. Let $\mathfrak{C}_v$ denote the set of cycles in $\mathfrak{C}$ incident with v and let G_v denote the union of the cycles in $\mathfrak{C}_v$. We note that any covering of G_v by noncontractible k-cycles must have at least $|\mathfrak{C}_v|$ elements, since otherwise $\mathfrak{C}$ is not a minimum covering of T by noncontractible k-cycles.

By Lemma 1, G_v has a covering $\mathfrak{C}'_v$ by noncontractible v-noncrossing k-cycles. By Lemma 3, $\mathfrak{C}'_v$ contains at most 3ξ distinct homotopy types. By a remark in the preceding paragraph, $|\mathfrak{C}'_v| \geqslant |\mathfrak{C}_v| \geqslant 3\xi(k-1)!\,(6k)^{k-1}$.

It follows that there is a subset $\mathfrak{C}''_v$ of $\mathfrak{C}'_v$ consisting of at least $(k-1)!\,(6k)^{k-1}$ pairwise homotopic cycles. Applying Lemma 6 to $\{C-v\colon C \in \mathfrak{C}''_v\}$, we get at least $6k$ of them have the same pairwise intersection. By Lemma 2, these common vertices are always in the same cyclic order, say $v=u_1, u_2, ..., u_t$.

All of these $6k$ cycles contain v and each has an edge not in any of the others. Therefore, either the $6k$ cycles are pairwise disjoint, except for v, or, by Lemma 5, there is an i such that from u_i to u_{i+1} we have at least $2k$ pairwise disjoint, pairwise homotopic paths from u_i to u_{i+1}.

In the first case, Lemma 5(3) implies at least $3k$ of the cycles are homotopic with v fixed. These $3k$ cycles can be ordered $C_1, ..., C_{3k}$ so that C_1 and C_{3k} bound a disc containing all the other curves. By Menger's theorem, there are $k-1$ pairwise disjoint paths across this disc from the vertices in $C_1 - v$ to the vertices in $C_{3k} - v$. These all have length at least $3k-1$ and the middle edge of any of these paths is not in any noncontractible cycle of length k in T. This is a contradiction.

In the second case, all the paths have the same length $l > 1$ and they can be ordered $P_1, ..., P_{2k}$ so that $P_1 \cup P_{2k}$ is a cycle that bounds a disc containing all the other paths. Again, Menger's theorem implies that there are $l-1$ paths across this disc from the vertices of $P_1 - \{u_i, u_{i+1}\}$ to the vertices of $P_{2k} - \{u_i, u_{i+1}\}$. A middle edge of any one of these paths is not in any noncontractible cycle of length k in T. This is a contradiction.

Finally, we consider the possibility that no vertex is in at least $3\xi(k-1)!\,(6k)^{k-1}$ cycles in $\mathfrak{C}$. Then there is a subset $\mathfrak{C}'$ of $\mathfrak{C}$ consisting of at least $|\mathfrak{C}|/3k!\,(6k)^{k-1} \geqslant 6\xi k$ pairwise disjoint cycles.

By Corollary 4.2, there must be at least $2k$ of these that are pairwise freely homotopic. Again, these can be ordered as $C_1, ..., C_{2k}$ so that C_1 and C_{2k} bound a cylinder containing the remaining freely homotopic cycles. Menger's theorem implies that there are k pairwise disjoint paths across this cylinder from $V(C_1)$ to $V(C_{2k})$. A middle edge of any of these paths is not in any noncontractible cycle of length k in T. This is the final contradiction that completes the proof of the Main Theorem. ∎

We conclude with some obvious relationships. An embedded graph G in a surface Σ is *r-representative* if every simple noncontractible curve in Σ meets G in at least r points. The embedded graph is *minor-minimal r-representative* if the deletion or contraction (in the surface) of each edge yields an embedded graph in Σ that is not r-representative.

It is clear that if G is an r-representative embedded graph in Σ, then there is an embedded graph H obtained from G by a sequence of edge deletions and contractions such that H is minor-minimal r-representation. Also, a triangulation T is r-representative if and only if the length of the shortest noncontractible cycle in T is at least r. Putting these two facts together shows that the size of a largest r-irreducible triangulation of Σ is not larger than the size of the largest minor-minimal r-representative embedding in Σ—only edge-deletion can be used to get from the triangulation to the minor-minimal embedding.

It is also easy to show that if G is a minor-minimal r-representative embedding in Σ $(r \geqslant 2)$, then putting a vertex in each face and joining it to all the vertices incident with the face and to new vertices in the middle of each edge in the boundary of the face yields a $(2r)$-irreducible triangulation T of Σ. (The case $r = 2$ is discussed in [MM] and the general case is discussed in [MN].) Since $|E(T) = 6\,|E(G)|$, we have the following result.

COROLLARY TO THE MAIN THEOREM. *Let G be a minor-minimal r-representative embedding in a surface of Euler genus ξ. Then $|E(G)| \leqslant r(2r)!\,(12r)^{2r}\,\xi^2$.*

We expect that the correct estimate for $|E(G)|$ in the corollary is $c_r \xi$ rather than $c_r \xi^2$. We also remark that a very short proof can be given for the Main Theorem (with the cruder estimate $|E(T)| \leqslant c_k \xi^k$), based on Corollaries 3.1, 3.2, Lemmas 4 and 6 (all easily proved). We have chosen the somewhat longer version in order to get the much improved bound of $c_k \xi^2$.

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